DHANAMANJURI UNIVERSITY

Examination- 2023 (Dec)

Three year course B.Sc. 5th Semester

Name of Programme	:	**B.Sc. Mathematics (Honours)**
Semester	:	**5th**
Paper Type	:	**DSE-II (Theory)**
Paper Code	:	**EMA-304**
Paper Title	:	**Number Theory**

Full Marks : 100

Pass Marks : 40 **Duration: 3 Hours**

The figures in the margin indicate full marks for the questions

Answer all the questions:

1. Write very short answers for the following questions: $\mathbf{1 \times 5 = 5}$

a) For what values of n, it has primitive roots.

b) Define Mobius function.

c) Define linear Diophantine equation in two variables.

d) Is 2 a quadratic residue of 5.

e) Define order of an integer 'a' modulo n.

2. Write short answers for the following questions: $\mathbf{3 \times 9 = 27}$

a) For each positive integer n, show that $\mu(n)\mu(n+1)\mu(n+2)\mu(n+3) = 0$

b) For each positive integer n, show that $n = \sum_{d|n} \Phi(d)$, where d runs through the positive divisors of n.

c) If $a|c$ and $b|c$ with $(a,b) = 1$, then prove that $ab|c$.

d) Verify that $1000!$ terminates in 249 zeros.

e) Evaluate the Legendre symbol

$\left(\frac{1234}{4567}\right)$

f) Find the units digit of 3^{100}.

g) If n has a primitive root then prove that it has exactly $\Phi(\Phi(n))$ of them.

h) Prove that for any odd prime p, the Legendre symbol $\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} (mod\ p)$, where a is an integer that is relatively prime to p.

i) Is the congruence $x^2 \equiv -46\ (mod\ 17)$ solvable? Justify.

3. Answer any three: **$6 \times 3 = 18$**

a) State and prove Chinese Remainder theorem.

b) Prove that given integers a and b not both of which are zero, there exists integers x and y such that $(a, b) = ax + by$.

c) Prove that if p is prime and $p \nmid a$, then $a^{p-1} \equiv 1\ (mod\ p)$. Is the converse true? Justify.

d) Prove that the linear congruence $ax \equiv b\ (mod\ n)$ has a solution iff $d|b$, where $(a, n) = d$. Also prove that if $d|b$, then it has d mutually incongruent solutions $mod\ n$.

4. Answer any three: **$6 \times 3 = 18$**

a) Prove that $\Phi - function$ is multiplicative.

b) State and prove Mobius Inversion formula.

c) If n and r are positive integers with $1 \leq r < n$, then prove that the binomial coefficient $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ is an integer. Also, prove that the product of any r consecutive positive integers is divisible by $r!$.

d) If $n = P_1^{K_1} P_2^{K_2} \dots P_r^{K_r}$ is the prime factorization of $n > 1$, then prove that

$\tau(n) = (K_1 + 1)(K_2 + 1) \ldots \ldots (K_r + 1)$ and

$$\sigma(n) = \frac{P_1^{K_1+1} - 1}{P_1 - 1} \frac{P_2^{K_2+1} - 1}{P_2 - 1} \ldots \ldots \frac{P_r^{K_r+1} - 1}{P_r - 1}$$

5. Answer any two: $\mathbf{2 \times 8 = 16}$

a) Prove that 2^K has no primitive roots for any integer $K \geq 3$.

b) State and prove Euler's Criterion.

c) If p is a prime number and $d|p-1$, then prove that the congruence $x^d - 1 = 0 \ (mod\ p)$ has exactly d solutions.

6. Answer any two: $\mathbf{2 \times 8 = 16}$

a) State and prove Quadratic Reciprocity law.

b) Let p be an odd prime and a an odd integer with $(a, p) = 1$, then $\left(\frac{a}{p}\right) = (-1)^{\sum_{k=1}^{\frac{p-1}{2}} \left[\frac{ka}{p}\right]}$.

c) Show that for any odd prime p,

$$\left(\frac{2}{P}\right) = \begin{cases} 1 & if\ p \equiv 1\ (mod\ 8)\ or\ p \equiv 7 (mod\ 8) \\ -1 & if\ p \equiv 3\ (mod\ 8)\ or\ p \equiv 5\ (mod\ 8) \end{cases}.$$

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