DHANAMANJURI UNIVERSITY

Examination- 2024 (Dec)

Four year course B.Sc./B.A. 5^{th} Semester

Name of Programme	**:**	**B.Sc./B.A. Mathematics(Honours)**
Paper Type	**:**	**CORE XIV{Theory}**
Paper Code	**:**	**CMA-314**
Paper Title	**:**	**Ring Theory and Linear Algebra-I**

Full Marks : 80

Pass Marks : 32 **Duration: 3 Hours**

The figures in the margin indicate full marks for the questions:
Answer all the question.

1. Choose and rewrite the correct answer: **$1 \times 3 = 3$**

i) Let R be the ring of all matrices of order 2 x 2 over integers and S be the set of 2 x 2 matrices of the form $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$ where a ,b are integers then

a) S is the left ideal b) S is the right ideal
c) S is both left and right ideal d) S is neither left and right ideal

ii) If $\mathbb{Q}, \mathbb{C}, \mathbb{Q}$ are the set of reals, complex numbers and the set of rational numbers, then which one of the following is not a vector space with usual notation ?

a) $\mathbb{R}(\mathbb{R})$ b) $\mathbb{C}(\mathbb{R})$
c) $\mathbb{R}(\mathbb{Q})$ d) $\mathbb{R}(\mathbb{C})$

iii) A linear transformation $T : V(F) \to W(F)$ is a non-singular if

a) $KerT = 0$
b) $RangeT = 0$
c) Rank T + Nullity T = $dimV(F)$
d) dim$V(F)$= dim $W(F)$.

2. Answer the following: $1 \times 6 = 6$

i) Give an example of a division ring which is not a field.

ii) Consider the ring $R = (0, 1, 2)$mod 3,find the characteristic of R.

iii) Show that $(1, i)$ forms a basis of $\mathbb{C}(\mathbb{R})$.

iv) Differentiate a spanning set form a basis of a Finite Dimensional Vector Space $V(F)$.

v) Define matrix of a linear transformation.

vi) For the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ as defined by $T(x, y, z) = (x, y)$.Determine ker T.

3.Answer the following: 3×5=15

i) Show that union of two ideals of a ring R need not be an ideal of R with example.

ii) Show that every field is an integral domain.

iii) If= $\{v_1, v_2, ..., v_n\}$ be a finite subset of a vector space which forms a basis of $V(F)$,show that $v \in V(F)$ has a unique representation.

iv) Show that intersection of two subspaces of a vector space $V(F)$ is again a subspace of $V(F)$.

v) Show that a linear transformation is a monomorphism iff it is onto.

4. Answer the following : 4×5=20

i) Show that a non empty subset S of a ring R is a subring of R iff $a, b \in S \implies ab, a - b \in S$.

ii) State and prove a necessary and sufficient condition for a non-empty subset $W(F)$ of a vector $V(F)$ to be a vector subspace.

iii) Let W be a subspace of a finite dimensional vector $V(F)$.Prove that $dim(\frac{v}{w}) = dimV - dimW$.

iv) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, 2x_1 + x_2 - x_1 - 2x_2 + 2x_3)$. Find the condition that (x_1, x_2, x_3) is in the range of T. Also show that Rank T = 2.

v) For the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (-y, -x)$.If $\beta = \{(1, 0), (0, 1)\}$ be the standard basis of $\mathbb{R}^2$ then find $[T]_{\beta\beta}$.

5.Answer any two: 6×2=12

i) Let R be a ring and I be an ideal of R,show that the quotient ring R/I from a ring.

ii) Prove that an ideal M of a commutative ring R is maximal iff R/M is a field.

iii) State and prove the fundamental theorem on Ring Homomorphism.

6.Answer any two: 6×2=12

i) Let W_1 and W_2 be two subspaces of a vector space $V(F)$. Prove that $V(F) = W_1 \oplus W_2 \iff V(F) = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$

ii) Prove that a finite dimensional vector space $V(F)$ has a basis.

iii) Suppose S is a finite subset of a finite dimensional vector space $V(F)$ such that $V(F) = L(S)$.Show that there exists a finite subset of S which forms a basis of $V(F)$.

7.Answer any two: 6×2=12

i) Find Range, Rank, Kernel and Nullity of the linear transformation $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x, y) = (x + y, y - x)$.

ii) State and Prove Sylvester's law on a linear transformation.

iii) Let V and W be two vector space of dimension m and n respectively.Show that $Hom(V, W)$ has dimension m,n.

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