DHANAMANJURI UNIVERSITY

Examination- 2023 (Dec)

Three year course B.Sc. 5th Semester

Name of Programme	**:**	**B.A/B.Sc. Mathematics (Honours)**
Semester	**:**	**5^{th}**
Paper Type	**:**	**Core-XII (Theory)**
Paper Code	**:**	**CMA-312**
Paper Title	**:**	**Group Theory-II**

Full Marks : 100

Pass Marks : 40 **Duration: 3 Hours**

The figures in the margin indicate full marks for the questions

Answer all the questions:

1. Answer each of the following: $\mathbf{1 \times 4 = 4}$

a) Let G be a group and f: G→ G be a mapping defined by $f(x) = x^{-1}$ ∀ x ∈ G. Then f is an automorphism if and only if

i) G is commutative
ii) G is non-commutative
iii) G is a finite cyclic group
iv) G≠ {e}, e is the identity element of G

b) The number of Sylow 2-subgroups of the symmetric group S_3 on 3-symbols is

i) 1
ii) 2
iii) 3
iv) Infinite

c) The number of distinct permutations on a finite group of order n is

i) n
ii) n^2
iii) ${}^{1}/_{2}\, n^2$
iv) $n!$

d) Let * :G X A → A be a group action, Ker(*) is

i) $\{x \mid x \in G$ s.t. $x*a=x \ \forall \ a \in A\}$

ii) $\{x \mid x \in G$ s.t. $x*a=a \ \forall \ a \in A\}$

iii) $\{x \mid x \in G$ s.t. $x*a=xa \ \forall \ a \in A\}$

iv) $\{x \mid x \in G$ s.t. $x*a=x^{-1}ax \ \forall \ a \in A\}$

2. Answer any four questions from the following: $1 \times 4 = 4$

a) Define an automorphism.

b) Write the condition under which the factor G/N, $N \trianglelefteq G$ is abelian.

c) Define an External Direct Product (E.D.P).

d) When is a group G said to act on a non-empty set A?

e) Write the statement of the fundamental theorem of finite abelian group.

f) Write the statement of the first Sylow's Theorem on p-groups.

3. Answer any four questions from the following: $2 \times 4 = 8$

a) Let G be a group and let H be a normal subgroup of G. Define a mapping

$f: G \rightarrow G/H$ by

$f(g) = Hg \ \forall \ g \in G$

Show that f is a homomorphism.

b) Let G be a cyclic group of order 4. Show that a mapping

$T: G \rightarrow G$ as defined by

$T(x) = x3 \forall \ x \in G$ is an automorphism

c) Let G be a group and let A be a non-empty set such that A=G. Define $*:GXA \rightarrow A$ by $g*a=gag^{-1} \forall \ g \in G, a \in A$ Show that * is a group action.

d) Show that G={$\pm$ 1, $\pm\, i$, $\pm$ j, $\pm$K} is a 2-group under multiplication as defined by

$$i^2 = j^2 = k^2 = -1$$

$$ij = j.i = K$$

$$jK = -Kj = i$$

$$Ki = -i.K = j$$

e) Prove that in a group G, $a^{o(G)}$=e for any a ∈ G where e is the identity element of G.

f) Let H and K be two subgroups of a finite group G. Define the double coset of H and K and write the formula for computing its order.

4. Answer any twelve questions from the following: $\mathbf{3 \times 12 = 36}$

a) Show that a homomorphism f:G →G' from a group G to another group G' is a monomorphism (i.e, 1-1) if and only if Ker f={e}, e being the identity element of G.

b) Show that I(G), the group of all inner automorphism is a normal subgroup of group Aut(G) of all automorphisms on G.

c) Let G act on A under * For a , b ∈ G define a relation a~b in A if there exists g ∈ G such that a= g*b. Show that ~ defines an equivalence relation and for a ∈ A,cl(a) is the orbit of a in G.

d) Prove that composite of two even permutations is again an even permutation.

e) Find fog and express it as composite of transpositions if f= (135) (24) and g=(245)(67).

f) Show that any sub group of a cyclic group is cyclic.

g) Let a, n (n≥1) be any two integers such that gcd (a,n)=1. Then prove that $a^{\Phi(n)} \equiv 1$(modn).

h) In the symmetric group S_3 on 3-symbols, find the orders of each element of S_3.

i) If H is the only Sylow p-subgroup of a group G then prove that H is normal in G and also conversely.

j) Let F be a group and H be a subgroup of G. Define C(H) and N(H) and show that they are non-empty subgroups of the group G.

k) Show that a group of order p^2, p being prime is abelian.

l) Let C be the set of complex numbers. Define a mapping $\Phi : C \to C$ by $\Phi(z)=\overline{z}$, $\overline{z}$ is the conjugate of $z \in C$. Prove that Φ is an endomorphism.

m) Show that a sub group of index 2 in a group G is a normal subgroup of the group.

n) If H_1 and H_2 are two normal subgroups of a group G such that $H_1 \sqsubseteq H_2$, then prove that H_2/H_1 is a normal subgroup of G/H_1.

5. Answer any eight questions from the following: $\mathbf{8 \times 6 = 48}$

a) If $f:G \to G'$ be an isomorphism from a group G onto a group G' then show that

   i) $f(e) = e'$, e and e' are identities of G and G'

   ii) $f(x^{-1}) = (f(x))^{-1} \ \forall\, x \in G$

   iii) $f(x^n) = (f(x))^n \ \forall\, n \in N, x \in G$

b) Let H and K be two subgroups of a group G such that H is normal in G then show that

$$HK/H \cong K/H \cap K$$

c) Let H_i, i=1,2,3,....n be normal subgroups of G for each i. Then prove that G is the IDP if and only if

   i) $G = H_1.H_2....H_n$

   ii) $H_i \cap (H_iH_2....H_{i-1}.H_{i+1}...H_n) = \{e\} \ \forall\, i=1,2,...n$

d) Let G be a group of order p^2, p being prime. Then show that either G is cyclic or G is isomorphic to the direct product of two cyclic groups each of order p.

e) Prove that $O(cl(a))=O(G)/O(N(a))\ \forall\ a \in G$.

f) Prove that $O(G)=O(Ga) \times O(G_a)$ where the symbols have their usual meanings.

g) Let G be a group and A be any non-empty set. Show that any homomorphism

$\Phi : G \rightarrow Sym(A)$ defines an action of G on A and conversely every action of G on A induces a homomorphism from $G \rightarrow Sym(A)$

h) Let G be a finite group and suppose p is prime such that $p \mid O(G)$ then prove that there exists $x \in G$ s.t. $O(x)=p$.

j) Prove that the number of Sylow p-subgroups of a group G is of the form 1+Kp where K is a positive integer and 1+Kp divides O(G).

k) Let G be a group and suppose G is the IDP of $H_1, H_2, \ldots H_n$. Let T be EDP of $H_1, H_2, \ldots H_n$. Show that $G \cong T$.

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