DHANAMANJURI UNIVERSITY

Examination - 2024 (June)

Four-Year Course BA/B.Sc. 4th Semester

Name of Programme	**:**	**B.A/B.Sc. Mathematics**
Paper Type	**:**	**Core-X(Theory)**
Paper Code	**:**	**CMA-210**
Paper Title	**:**	**Riemann Integration and Series of Functions**
Full Marks	**:**	**40**
Pass Marks	**:**	**16**
Duration	**:**	**2 Hours**

The figures in the margin indicate full marks for the questions
Answer the following questions:

1. Choose and rewrite the correct answer for each of the following questions: **$1 \times 3 = 3$**

a) If $f(x) = x - 6$ and $P = \{0,3,4,6\}$is a partition of $[0,6]$, then the value of the oscillatory sum $W(P, f)$ is

i) 7
ii) 14
iii) 26
iv) 32

b) The number of points of infinite discontinuity of the improper integral $\int_3^5 \frac{x^2}{\sqrt{x^2-8x+15}}dx$ is

i) 1
ii) 2
iii) 3
iv) 4

c) The power series $1+2x+3x^2+4x^3+\cdots$, has radius of convergence

i) 0
ii) 1
iii) 2
iv) ∞

2. Write very short answer on any five from the following questions: **1 × 5 = 5**

a) Give an example of bounded function which is not Riemann integrable.

b) State fundamental theorem of integral calculus.

c) Examine the convergence of improper integral $\int_0^\infty \frac{1}{1+x^2}dx$

d) State Dirichlet's test for convergence of improper integral.

e) When is a sequence of functions $\{f_n\}$ said to be uniformly convergence in an interval [a, b]?

f) State Weierstrass M-Test (for uniform convergence).

3. Write short answer on any two from the following: **2 × 3 = 6**

a) If P^* is a refinement of a partition P for a bounded function f on [a, b], then prove that $L(P^*, f) \geq L(P, f)$

b) Give reason for R-integrability of the function $f(x) = x[2x]$ in [0, 2], where [x] denotes the greatest integer not greater than x and evaluate $\int_0^2 f(x)dx$.

c) Examine the convergence of improper integral $\int_0^1 \log x^4 dx$.

d) Show that $\int_0^\infty \frac{1}{(1+x)\sqrt{x}}dx$ is convergent with value π.

e) Test for uniform convergence the sequence of functions $\{f_n\}$, where $f_n = \frac{nx}{1+n^2x^2}, \forall x \in \mathbf{R}$

4. Write short answer on any two from the following: $2 \times 4 = 8$

a) Let $f(x) = \sin x$ and $P = \left\{0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{6}\right\}$ be a partition of interval $\left[0, \frac{5\pi}{6}\right]$.
Find:

i) $||P||$
ii) $L(P, f)$
ii) $U(P, f)$

(The symbols have their usual meaning)

b) Prove that every continuous function on [a, b] is integrable on [a, b]

c) Show that the improper integral $\int_a^b \frac{1}{(x-a)^p}dx$ converges at a if and only if $p < 1$

d) Show that the sequence $\{f_n\}$, where $f_n(x) = \tan^{-1} nx$, $x \geq 0$ is uniformly convergent in any interval [a, b], a > 0, but is only pointwise convergent in [0, b].

e) If R is the radius of convergence of power series $\sum a_n x^n$, then show that the radius of convergence of power series $\sum n a_n x^{n-1}$ and $\sum \frac{a_n}{n+1} x^{n+1}$ is also R.

5. Answer any one from the following questions: $6 \times 1 = 6$

a) Let $|f(x)| \leq K, \forall x \in [a, b]$ and P be a partition of [a, b] with norm $\leq \delta$. If P^* is a refinement of P containing just one more point than P, then prove that $U(P^*, f) \leq U(P, f) \leq U(P^*, f) + 2K\delta$.

b) State a necessary and sufficient condition for integrability of a bounded function f on an interval $[a, b]$ and prove the same.

c) If f_1 and f_2 are both bounded and integrable functions on $[a, b]$, then prove that $f = f_1 + f_2$ is also bounded and integrable on $[a, b]$ and
$\int_a^b f_1(x)dx + \int_a^b f_2(x)dx = \int_a^b f(x)dx.$

6. Answer any one from the following questions: $6 \times 1 = 6$

a) Test the convergence of the improper integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$.

b) Show that $\int_0^\infty \frac{dx}{(1+x^2)^2}$ is convergent.

7. Answer any one from the following questions: $6 \times 1 = 6$

a) State Cauchy's Criterion for uniform convergence. Also prove the same.

b) Determine the radius of convergence and the exact interval of convergence of the power series $\sum \frac{n+1}{(n+2)(n+3)}x^n$.

c) State and prove Abel's theorem for power series.