DHANAMANJURI UNIVERSITY
JUNE - 2023

Name of Programme	**:**	**B.Sc. Mathematics (Honours)**
Semester	**:**	**4^{th}**
Paper Type	**:**	**Core-IX (Theory)**
Paper Code	**:**	**CMA-209**
Paper Title	**:**	**Ring Theory and Linear Algebra I**
Full Mark	**:**	**50**
Pass Mark	**:**	**20**
Duration	**:**	**2 Hours**

The figures in the margin indicate full marks for the questions.
Answer only 5 (five) from the following questions:

1. Define Rank and Nullity of a linear transformation T. Also state and prove Rank-Nullity theorem. 2+8=10

2. Define ordered basis of a finite dimensional vector space. 1
   Let T be a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ defined by $T(x, y, z) = (x + y, 2z - x)$.
   i) If β and β' are standard ordered basis for $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively, find $[T]_{\beta,\beta'}$. 4
   ii) If $\beta = \{(1, 0, -1), (1, 1, 1), (1, 0, 0)\}$ and $\beta' = \{(0, 1), (1, 0)\}$ are ordered basis for $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively, find $[T]_{\beta,\beta'}$. 5

3. If M and N are two ideals of a ring R, then prove that $\frac{M+N}{M} \cong \frac{N}{M\cap N}$. Also find all the six ring homomorphisms from $\mathbb{Z}_{12} \to \mathbb{Z}_{30}$. 6+4=10

4. Prove that a basis of a vector space is the maximal linearly independent set, and conversely. 10

5. i) Let W be a subspace of a finite dimensional vector space V. Then prove that $\dim V = \dim W + \dim V/W$. 7
   ii) Show that the vectors $\{(1, 1, 2), (1, 2, 5), (5, 3, 4)\}$ are linearly dependent in $\mathbb{R}^3$. 3

6. Define subspace of a vector space. Let V be the vector space of all functions from $\mathbb{R} \to \mathbb{R}$. Let $V_e = \{f \in V | f \text{ is even}\}$ and $V_o = \{f \in V | f \text{ is odd}\}$. Then prove that V_e and V_o are subspaces of V, and $V = V_e \oplus V_o$. 10

7. Define prime ideal of a ring. Also prove that an ideal P of a commutative ring R is a prime ideal of R iff $\frac{R}{P}$ is an integral domain. 10

8. i) If A and B are two ideals of a ring R, then prove that $A + B$ is an ideal containing both A and B. Also show that $A + B = \langle A \cup B\rangle$. 5
   ii) If A is an ideal of a ring R with unity such that $1 \in A$. Then show that $A = R$. 5

9. i) Prove that a finite commutative ring without zero divisors is a field. 5
   ii) Prove that every ideal is a subring. Is the converse true? Justify. 5

10. i) Let S be a subset of a vector space V. Prove that the linear span $L(S)$ of S is the smallest subspace of V containing S. 5
    ii) Is the transformation $T : \mathbb{R} \to \mathbb{R}^3$ defined by $T(x) = (x, x^2, x^3)$ linear. 2
    iii) Show that in a vector space $V(F)$:
      i. $\alpha v_1 = \alpha v_2 \implies v_1 = v_2$ ($\alpha \neq 0$). 2
      ii. $\alpha v = 0, \alpha \neq 0 \implies v = 0$. 1