DHANAMANJURI UNIVERSITY

Four-Year Course BA/B.Sc. 3rd Semester

DECEMBER-2022

Name of Programme	**:**	**B.A/B.Sc Mathematics (Honours)**
Paper Type	**:**	**Core-5(Theory)**
Paper Code	**:**	**CMA-205**
Paper Title	**:**	**Theory of Real Functions**
Full mark	**:**	**100**
Pass Mark	**:**	**40**
Duration	**:**	**3 Hours**

The figures in the margin indicate full marks for the questions.
Answer all the questions:

1. Give the definitions of the limits: 4

   (a) $\lim_{x\to 1} f(x) = 1$ and

   (b) $\lim_{z\to\infty} f(x) = 1$

2. Using the $\varepsilon - \delta$ definition, show that $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$. 4

3. Show that $\lim_{z\to 4} \frac{1}{(x-4)^2} = \infty$. 4

4. Prove that the function $f(x) = \sin x$ is uniformly continuous on $[0,\infty[$. 4

5. Prove that a function which is uniformly continuous on an interval is continuous on that interval. 4

6. Using Taylor's Theorem with $n = 2$, approximate $\sqrt{1+x}$, $x > -1$. 4

Or

Expand the function $\sin x$ in powers of x in a finite series with the Lagrange form of remainder. 4

7. Show that the function defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} 1, & \text{where x is irrational} \\ -1, & \text{where x is rational} \end{cases}$$

is discontinuous at every point. 6

8. Let $f : I \to \mathbb{R}$ be differentiable on the interval I. Then, prove that f is increasing on I if and only if $f'(x) \geq 0$ for all $x \in I$. 6

9. State Cauchy's necessary and sufficient condition for the existence of a limit. Hence, show that $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist. 6

10. If a function f is continuous on a closed interval $[a,b]$ and $f(a)$ and $f(b)$ are of opposite signs, then prove that there exists at least one point $\alpha \in]a,b[$ such that $f(\alpha) = 0$. 8

Or

Prove that if a function is continuous in a closed interval, then it is bounded therein. 8

11. Prove that a function f defined on an interval I is continuous at a point $c \in I$ if and only if for every sequence $\{c_n\}$ in I converging to c, $\lim_{n \to \infty} f(c_n) = f(c)$. 8

12. State and prove Caratheodory's Theorem. 8

13. State and prove Rolle's Theorem. Also, write the geometrical interpretation of the theorem. 8

14. Let $I \subseteq \mathbb{R}$ be an interval, let $f : I \to \mathbb{R}$, let $c \in I$, and assume that f has a derivative at c . Then, prove that: 8

    a) If $f'(c) > 0$, then there is a number $\delta > 0$ such that $f(x) > f(c)$ for $x \in I$ such that $c < x < c + \delta$.
    b) If $f'(c) < 0$, then there is a number $\delta > 0$ such that $f(x) < f(c)$ for $x \in I$ such that $c - \delta < x < c$.

Or

State and prove Darboux's Theorem.

15. State and prove Taylor's Theorem in finite form with Lagrange form of remainder. 8

16. Let I be an open interval and let $f : I \to \mathbb{R}$, have a second derivative on I. Then, prove that f is a convex function on I if and only if $f''(x) \geq 0$ for all $x \in I$. 8

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